Mark Scheme (Results)

Summer 2018

Pearson Edexcel GCE
In Further Pure Mathematics 3 (6669/01)

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

Question Number	Scheme	Notes	Marks
1(a)	$\left(\tanh x = \dfrac{\sinh x}{\cosh x}\right) = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$ or $\dfrac{\frac{e^{2x} - 1}{2e^x}}{\frac{e^{2x} + 1}{2e^x}}$	Substitutes the correct exponential forms. Note that the $\tanh x = \dfrac{\sinh x}{\cosh x}$ may be implied.	M1
	$= \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ *	Correct proof with no errors or omissions or notational errors such as using sin for sinh	A1*
	Note that the question says "starting from the definitions of sinh*x* and cosh*x* in terms of exponentials" so: $\boxed{\sinh x = \dfrac{e^x - e^{-x}}{2},\ \cosh x = \dfrac{e^x + e^{-x}}{2},\ \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}}$ scores M1A1 **BUT** $\boxed{\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}}$ scores M0A0 as sinh*x* and cosh*x* have not been defined		
			(2)
(b)	$y = \operatorname{artanh}\theta \Rightarrow \tanh y = \theta \Rightarrow \theta = \dfrac{e^{2y} - 1}{e^{2y} + 1}$ $\theta\left(e^{2y} + 1\right) = e^{2y} - 1 \Rightarrow e^{2y}(\theta - 1) = -1 - \theta \Rightarrow e^{2y} = \dfrac{1 + \theta}{1 - \theta}$ M1 for setting $\theta = \dfrac{e^{2y} - 1}{e^{2y} + 1}$ or any other variables for θ and *y* e.g. $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ and uses correct processing (allow sign errors only) to make $e^{2\text{"}y\text{"}}$ or $e^{\text{"}y\text{"}}$ the subject		M1
	$e^{2y} = \dfrac{1 + \theta}{1 - \theta} \Rightarrow 2y = \ln\left(\dfrac{1 + \theta}{1 - \theta}\right)$	Removes e correctly by taking ln's. **Dependent on the first method mark.**	**d**M1
	$y = \dfrac{1}{2}\ln\left(\dfrac{1 + \theta}{1 - \theta}\right)$ * or $\dfrac{1}{2}\ln\dfrac{1 + \theta}{1 - \theta}$ * Correct completion with no errors. Must be in terms of θ for this mark but allow "mixed" variables for the M's. This mark should be withheld if there are any errors such as the appearance of a "tan" instead of "tanh" and/or missing variables. The proof does need to convey that $\operatorname{artanh}\theta = \dfrac{1}{2}\ln\left(\dfrac{1 + \theta}{1 - \theta}\right)$ So if *y* has been defined as artanhθ and the proof ends $y = \dfrac{1}{2}\ln\left(\dfrac{1 + \theta}{1 - \theta}\right)$, this is acceptable. So must be in terms of θ for the A mark but allow other variables to be used for the M's. Allow arctanh, artanh, $\tanh^{-1}$ etc. for the inverse		A1*
			(3)

	(b) Alternative:		
	$\text{artanh}\,\theta = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right) \Rightarrow \theta = \tanh\left(\frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)\right)$		
	$\theta = \dfrac{e^{\ln\left(\frac{1+\theta}{1-\theta}\right)}-1}{e^{\ln\left(\frac{1+\theta}{1-\theta}\right)}+1}$	Uses part (a) to express θ in terms of e	M1
	$= \dfrac{\frac{1+\theta}{1-\theta}-1}{\frac{1+\theta}{1-\theta}+1} = \dfrac{1+\theta-1+\theta}{1+\theta+1-\theta} = \theta$	Removes e's and ln's **correctly**. **Dependent on the first method mark.**	**d**M1
	$\Rightarrow \text{artanh}\,\theta = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)$* Allow $\frac{1}{2}\ln\frac{1+\theta}{1-\theta}$*	Obtains $\theta = \theta$ with no errors and makes a conclusion. Must be in terms of θ for this mark but allow a different variable for the M's. This mark should be withheld if there are any errors such as the appearance of a "tan" instead of "tanh" and/or missing variables.	A1*
			(3)
	Attempts that assume $\text{artanh}\,\theta = \dfrac{\text{arsinh}\,\theta}{\text{arcosh}\,\theta}$ score no marks in (b)		
			Total 5

Question Number	Scheme	Notes	Marks
2	$y = 5\cosh x - 6\sinh x$		
(a)	$5\cosh x - 6\sinh x = 0 \Rightarrow 5\left(\frac{e^x + e^{-x}}{2}\right) - 6\left(\frac{e^x - e^{-x}}{2}\right) = 0$ Substitutes the correct exponential forms but allow the "2's" to be missing		M1
	$e^{2x} = 11$	Correct equation	A1
	$x = \ln\sqrt{11}$	Correct value (oe e.g. $\frac{1}{2}\ln 11$)	A1
	Alternative 1		
	$5\cosh x - 6\sinh x = 0 \Rightarrow \tanh x = \frac{5}{6}$	Rearranges to tanh$x$ = …	M1
	$x = \text{artanh}\left(\frac{5}{6}\right)$	Correct equation	A1
	$x = \ln\sqrt{11}$	Correct value (oe e.g. $\frac{1}{2}\ln 11$)	A1
	Alternative 2		
	$5\cosh x - 6\sinh x = 0 \Rightarrow 25\cosh^2 x = 36\sinh^2 x$ $25(1+\sinh^2 x) = 36\sinh^2 x$ or $25\cosh^2 x = 36(\cosh^2 x - 1)$ $\sinh^2 x = \frac{25}{11}$ or $\cosh^2 x = \frac{36}{11}$ Rearranges to sinh^2x = … or cosh^2x = …		M1
	$\Rightarrow \sinh x = (\pm)\frac{5}{\sqrt{11}}$ or $\Rightarrow \cosh x = (\pm)\frac{6}{\sqrt{11}}$	Correct equation (Allow ±)	A1
	$x = \ln\sqrt{11}$	Correct value (oe e.g. $\frac{1}{2}\ln 11$)	A1
	Note that this is not a proof so allow "h's" to be lost along the way as long as the intention is clear.		
			(3)

(b)	$(5\cosh x - 6\sinh x)^2 \equiv 25\cosh^2 x - 60\cosh x\sinh x + 36\sinh^2 x$ $\equiv 25\left(\frac{\cosh 2x+1}{2}\right) - 60\frac{1}{2}\sinh 2x + 36\left(\frac{\cosh 2x-1}{2}\right)$ Squares to obtain $p\cosh^2 x + q\cosh x\sinh x + r\sinh^2 x,\quad p,q,r \neq 0$ and attempts to use at least one **correct** "double angle" hyperbolic identity for cosh2x or sinh2x e.g. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1, \sinh 2x = 2\sinh x\cosh x$		M1
	$= \frac{61}{2}\cosh 2x - 30\sinh 2x - \frac{11}{2}$	Two correct terms in their final expression	A1
		All correct terms in their final expression	A1
			(3)
	Alternative 1 for (b) using exponentials after squaring:		
	$(5\cosh x - 6\sinh x)^2 \equiv 25\cosh^2 x - 60\cosh x\sinh x + 36\sinh^2 x$ $= 25\left(\frac{e^x+e^{-x}}{2}\right)^2 - 60\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right) + 36\left(\frac{e^x-e^{-x}}{2}\right)^2$ $= \left(\frac{25}{2}+\frac{36}{2}\right)\left(\frac{e^{2x}+e^{-2x}}{2}\right) - 30\left(\frac{e^{2x}-e^{-2x}}{2}\right) + \frac{50}{4} - \frac{72}{4}$ $= (...)(\cosh 2x) + (...)(\sinh 2x) + (...)$ Squares to obtain $p\cosh^2 x + q\cosh x\sinh x + r\sinh^2 x,\quad p,q,r \neq 0$ and attempts to use at least one correct exponential definition for cosh2x or sinh2x		M1
	$= \frac{61}{2}\cosh 2x - 30\sinh 2x - \frac{11}{2}$	Two correct terms in their final expression	A1
		All correct terms in their final expression	A1
			(3)
	Alternative 2 for (b) using exponentials before squaring:		
	$(5\cosh x - 6\sinh x)^2 = \left(5\left(\frac{e^x+e^{-x}}{2}\right) - 6\left(\frac{e^x-e^{-x}}{2}\right)\right)^2 = \left(\frac{11}{2}e^{-x} - \frac{1}{2}e^x\right)^2$ $= \left(\frac{121}{4}e^{-2x} + \frac{1}{4}e^{2x} - \frac{11}{2}\right)$ $= (...)(\cosh 2x) + (...)(\sinh 2x) + (...)$ Substitutes the correct exponential forms and squares to obtain $pe^{-2x} + qe^{2x} + r,\quad p,q,r \neq 0$ and attempts to use at least one correct exponential definition for cosh2x or sinh2x		M1
	$= \frac{61}{2}\cosh 2x - 30\sinh 2x - \frac{11}{2}$	Two correct terms in their final expression	A1
		All correct terms in their final expression	A1
			(3)

(c)	**Note that π is not needed for the first 3 marks of (c)**		
	$V=(\pi)\int\left(\frac{61}{2}\cosh 2x-30\sinh 2x-\frac{11}{2}\right)\mathrm{d}x$	Uses $V=(\pi)\int y^2\ \mathrm{d}x$ with their y^2 where y^2 is of the form $=a\cosh 2x+b\sinh 2x+c$	M1
	$(\pi)\left[\frac{61}{4}\sinh 2x-15\cosh 2x-\frac{11}{2}x\right]$	Correct integration, ft their a, b and c **or** the letters a, b and c **or** a combination of both **or** "made up" values.	A1ft
	$(\pi)\left[\frac{61}{4}\sinh(\ln 11)-15\cosh(\ln 11)-\frac{11}{4}(\ln 11)-(-15)\right]$ Note that $\cosh(\ln 11)=\frac{61}{11}$, $\sinh(\ln 11)=\frac{60}{11}$ Correct use of limits. Must see 0 **and** their value from (a) substituted into all 3 terms (although the "0's" can be implied) and subtracted the right way round. **Dependent on the first method mark.**		**d**M1
	$=\left(15-\frac{11}{4}\ln 11\right)\pi$ or e.g. $\left(15-\frac{11}{2}\ln\sqrt{11}\right)\pi$ Or e.g. $\frac{30\pi}{2}-\frac{11\pi}{4}\ln 11$, $15\pi-\frac{11\pi}{2}\ln\sqrt{11}$	Correct exact answer in any equivalent **exact** form.	A1
			(4)
	Alternative to (c) using exponentials:		
	$V=\frac{(\pi)}{4}\int\left(121\mathrm{e}^{-2x}-22+\mathrm{e}^{2x}\right)\mathrm{d}x$	Uses $V=(\pi)\int y^2\ \mathrm{d}x$	M1
	$\frac{(\pi)}{4}\left[\frac{\mathrm{e}^{2x}}{2}-\frac{121\mathrm{e}^{-2x}}{2}-22x\right]$	Correct integration. You can follow through their expansion from part (a).	A1ft
	$\frac{(\pi)}{4}\left[\frac{11}{2}-\frac{11}{2}-11\ln 11-\frac{1}{2}+\frac{121}{2}\right]$	Correct use of limits (0 and their value from (a)). **Dependent on the first method mark.**	**d**M1
	$=\left(15-\frac{11}{4}\ln 11\right)\pi$ or e.g. $\left(15-\frac{11}{2}\ln\sqrt{11}\right)\pi$ Or e.g. $\frac{30\pi}{2}-\frac{11\pi}{4}\ln 11$, $15\pi-\frac{11\pi}{2}\ln\sqrt{11}$	Correct exact answer in any equivalent **exact** form.	A1
			(4)
			Total 10

Question Number	Scheme	Notes	Marks
3	$\mathbf{M}=\begin{pmatrix}3 & k & 2\\ -1 & 0 & 1\\ 1 & k & 1\end{pmatrix}$		
(a)	$\begin{vmatrix}3-3 & k & 2\\ -1 & -3 & 1\\ 1 & k & 1-3\end{vmatrix}=(3-3)\left[-3(1-3)-k\right]-k\left[-1(1-3)-1\right]+2(-k+3)\ \ (=-3k+6)$ Attempts determinant of $\mathbf{M}-3\mathbf{I}$ or $\begin{vmatrix}3-\lambda & k & 2\\ -1 & -\lambda & 1\\ 1 & k & 1-\lambda\end{vmatrix}=(3-\lambda)\left[-\lambda(1-\lambda)-k\right]-k\left[-1(1-\lambda)-1\right]+2(-k+\lambda)$ $\left(=-\lambda^3+4\lambda^2-\lambda-3k\right)$ $\lambda=3\Rightarrow \det(\mathbf{M}-\lambda\mathbf{I})=(3-3)\left[-3(1-3)-k\right]-k\left[-1(1-3)-1\right]+2(-k+3)$ Attempts determinant of $\mathbf{M}-\lambda\mathbf{I}$ and substitutes $\lambda=3$		M1
	Should be a recognisable attempt at the determinant (if there is any doubt at least 2 "terms" should be correct)		
	$0-k+2(-k+3)=0\Rightarrow k=2$	Puts = 0 (may be implied) and solves for k. **Dependent on the first M**	**d**M1
		$k=2$	A1
			(3)
	Alternative to part (a):		
	$\begin{pmatrix}3 & k & 2\\ -1 & 0 & 1\\ 1 & k & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$		
	$3x+ky+2z=3x$ $-x+z=3y$ $x+ky+z=3z$	Forms 3 equations using the eigenvalue 3	M1
	Let e.g. $x=1$ $\Rightarrow ky=-2z,\ z=3y+1,\ ky+1=2z$ $\Rightarrow 4z=1\Rightarrow z=\frac{1}{4},\ y=-\frac{1}{4}$ $\Rightarrow k=...$ Or e.g. $\Rightarrow ky=-2z,\ ky=2z-x, z=3y+x$ $\Rightarrow ky=-6x-2x,\ 2ky=12y+2x\Rightarrow 3ky=6y$ $\Rightarrow k=...$	Allocates a non-zero value to one of x or y or z and solves for the other two variables and finds a value for k Or Solves to obtain a value for k **Dependent on the first M**	**d**M1
	$\Rightarrow k=2$		A1
			(3)

(b)	$\det(\mathbf{M}-\lambda\mathbf{I})=(3-\lambda)\left[-\lambda(1-\lambda)-k\right]-k\left[-1(1-\lambda)-1\right]+2(-k+\lambda)$ Attempts determinant of $\mathbf{M}-\lambda\mathbf{I}$ (may be seen in (a)) but must be seen or used in (b) to score in (b)		M1
	$\det(\mathbf{M}-\lambda\mathbf{I})=(3-\lambda)\left[-\lambda(1-\lambda)-2\right]-2\left[-1(1-\lambda)-1\right]+2(-2+\lambda)=0$ Uses their *k* in their determinant and puts = 0 (May be implied by their work) **Dependent on the first M**		**d**M1
	$\{(3-\lambda)\}(\lambda^2-\lambda-2)=0\Rightarrow\lambda=...$	Solves 3TQ to find the 2 other eigenvalues (apply usual rules if necessary). **Dependent on both previous M's**	**dd**M1
	If they multiply out they should get $\lambda^3-4\lambda^2+\lambda+6=0$ and may use a calculator to obtain $\lambda=-1,\ 2\ (\text{and } 3)$		
	$\lambda=-1,\ 2$	Correct eigenvalues. (Must follow *k* = 2)	A1
			(4)
(c)	$\begin{pmatrix}3 & "2" & 2\\ -1 & 0 & 1\\ 1 & "2" & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Rightarrow\begin{matrix}3x+"2"y+2z=3x\\ -x+z=3y\\ x+"2"y+z=3z\end{matrix}$ or $\begin{pmatrix}0 & "2" & 2\\ -1 & -3 & 1\\ 1 & "2" & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{matrix}"2"y+2z=0\\ -x-3y+z=0\\ x+"2"y-2z=0\end{matrix}$ Expands to obtain at least 2 equations. Allow if *k* is present.		M1
	$k\begin{pmatrix}4\\ -1\\ 1\end{pmatrix}$ or $k(4\mathbf{i}-\mathbf{j}+\mathbf{k})$	Any non-zero multiple but must be a vector	A1
	Note that the vector product of any 2 rows of **M - 3**I** also gives an eigenvector**		
			(2)
			Total 9

Note on Determinants:

Note that determinants can be found using any row or column

And also by applying the rule of Sarrus which is:

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

Please look out for these alternative approaches in Question 3

Question Number	Scheme	Notes	Marks
4	$y = \text{arsinh}x + x\sqrt{x^2+1}, \quad 0 \le x \le 1$		
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}$	$\frac{\mathrm{d}(\text{arsinh}x)}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}}$	B1
		$\frac{\mathrm{d}\left(x\sqrt{x^2+1}\right)}{\mathrm{d}x} = \frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}$	B1
	E.g. $= \frac{1+x^2+1+x^2}{\sqrt{x^2+1}} = \ldots$ or $= \frac{1+x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} = \ldots$	Processes **3 terms** of the form $\frac{A}{\sqrt{x^2+1}}, \frac{Bx^2 \text{ or } Bx}{\sqrt{x^2+1}}, C\sqrt{x^2+1}$ using correct algebra (allow sign slips only) to obtain a single term.	M1
	$= 2\sqrt{x^2+1}$ *	cso Allow $2\left(x^2+1\right)^{\frac{1}{2}}$	A1
			(4)
(b)	$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + 4\left(x^2+1\right)$	Attempts $\int \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ **with the printed answer** from part (a) (limits not needed here) but must see a step before the given answer.	M1
	$\Rightarrow (L=)\int_0^1 \sqrt{5+4x^2}\ \mathrm{d}x$ *	Answer **as printed** with no errors including limits and "dx" Allow $\int_0^1 \sqrt{4x^2+5}\ \mathrm{d}x$	A1*
			(2)

(c)	$x=\frac{\sqrt{5}}{2}\sinh u \Rightarrow \frac{dx}{du}=\frac{\sqrt{5}}{2}\cosh u$		
	$\Rightarrow L=\int \sqrt{5+5\sinh^2 u}\ \frac{\sqrt{5}}{2}\cosh u(du)$	Fully substitutes into $\int \sqrt{4x^2+5}\ dx$	M1
	$=\frac{5}{2}\int \cosh^2 u(du)$	Correct integral including the 5/2. Allow e.g. $\frac{5}{2}\int \cosh u \cosh u(\ du)$	A1
	$=\frac{5}{4}\int (\cosh 2u+1)(\ du)$	Applies $\cosh 2u=\pm 2\cosh^2 u\pm 1$ to an integral of the form $k\int \cosh^2 u\, du$. **Dependent on the first method mark.**	**d**M1
	$=\frac{5}{4}\left[\frac{1}{2}\sinh 2u+u\right]$	Correct integration: $k(\cosh 2u+1)\rightarrow k\left(\frac{1}{2}\sinh 2u+u\right)$	A1
	$=[\ldots\ldots]_0^{\operatorname{arsinh}\frac{2}{\sqrt{5}}\left(\text{or}\ln\sqrt{5}\right)}$ Note $\frac{1}{2}\sinh\left(2\left(\operatorname{arsinh}\frac{2}{\sqrt{5}}\right)\right)=\frac{6}{5}$	Use of correct limits or returns to x and uses 0 and 1. The use of 0 may be implied. **Dependent on both method marks.**	**dd**M1
	$=\frac{3}{2}+\frac{5}{8}\ln 5$	Allow equivalent exact answers. E.g. $\frac{3}{2}+\frac{5}{4}\ln\sqrt{5},\ \ \frac{3}{2}+\frac{5}{4}\ln\left(\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{5}}\right)$	A1
	May need to check their answer and could be implied by awrt 2.51 if their integration is correct. If the integration is incorrect and no substitution is shown, you may need to check their answer, but score M0 if the answer does not follow.		
			(6)
	Note that the variable may change mid-solution once the substitution has been made e.g. $u \rightarrow x$ but this should not be penalised unless there is a clear error in the solution		

	Note that having reached $\frac{5}{2}\int \cosh^2 u \, du$, candidates may use exponentials. Score the last 4 marks in (b) as follows:		
	$\frac{5}{2}\int \cosh^2 u \, du = \frac{5}{8}\int \left(e^{2u} + 2 + e^{-2u}\right) du$	Uses $\cosh u = \frac{1}{2}\left(e^u + e^{-u}\right)$ and squares applies to an integral of the form $k\int \cosh^2 u \, du$	**d**M1
	$= \frac{5}{8}\left[\frac{1}{2}e^{2u} + 2u - \frac{1}{2}e^{-2u}\right]$	Correct integration	A1
	$= [......]_0^{\text{arsinh}\frac{2}{\sqrt{5}}\left(\text{or} \ln\sqrt{5}\right)}$	Use of correct limits or returns to x and uses 0 and 1. **Dependent on both method marks.**	**dd**M1
	$= \frac{3}{2} + \frac{5}{8}\ln 5$	Allow equivalent exact answers. E.g. $\frac{3}{2} + \frac{5}{4}\ln\sqrt{5}$, $\frac{3}{2} + \frac{5}{4}\ln\left(\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}\right)$	A1
	May need to check their answer and could be implied by awrt 2.51 if their integration is correct. If the integration is incorrect and no substitution is shown, you may need to check their answer, but score M0 if the answer does not follow.		
			(6)
			Total 12

Question Number	Scheme	Notes	Marks
5	$I_n = \int x^n \sqrt{(x+8)}\,\mathrm{d}x$		
(a)	$I_n = \frac{2}{3}x^n(x+8)^{\frac{3}{2}} - \int \frac{2}{3}nx^{n-1}(x+8)^{\frac{3}{2}}(\mathrm{d}x)$	Parts in the correct direction	M1
		Correct expression	A1
	$I_n = \ldots - \frac{2}{3}n\int x^{n-1}(x+8)(x+8)^{\frac{1}{2}}(\mathrm{d}x)$	Writes $(x+8)^{\frac{3}{2}}$ as $(x+8)(x+8)^{\frac{1}{2}}$	M1
	$I_n = \frac{2}{3}x^n(x+8)^{\frac{3}{2}} - \frac{2}{3}nI_n - \frac{16}{3}nI_{n-1}$	Substitutes I_n and I_{n-1} correctly. **Dependent on the previous M mark**	**d**M1
	$I_n + \frac{2}{3}nI_n = \frac{2}{3}x^n(x+8)^{\frac{3}{2}} - \frac{16}{3}nI_{n-1}$	Collects I_n terms to lhs. **Dependent on both previous M marks**	**dd**M1
	$I_n = \frac{2x^n(x+8)^{\frac{3}{2}}}{2n+3} - \frac{16n}{2n+3}I_{n-1}$	All correct	A1
			(6)

	$I_0=\int\sqrt{(x+8)}\,\mathrm{d}x=\frac{2}{3}(x+8)^{\frac{3}{2}}(+c)$	Attempts I_0 (must be of the form $k(x+8)^{\frac{3}{2}}$)	M1
		Correct expression	A1
	The first 2 marks may be implied by $\frac{76\sqrt{2}}{3}$		
	$I_2=\frac{2x^2(x+8)^{\frac{3}{2}}}{2(2)+3}-\frac{16(2)}{2(2)+3}I_1$ **or** $I_1=\frac{2x(x+8)^{\frac{3}{2}}}{2(1)+3}-\frac{16(1)}{2(1)+3}I_0$	Reduction formula applied at least once	M1
	$I_2=\frac{2x^2(x+8)^{\frac{3}{2}}}{2(2)+3}-\frac{16(2)}{2(2)+3}I_1$ **and** $I_1=\frac{2x(x+8)^{\frac{3}{2}}}{2(1)+3}-\frac{16(1)}{2(1)+3}I_0$ $I_2=\frac{2x^2(x+8)^{\frac{3}{2}}}{2(2)+3}-\frac{16(2)}{2(2)+3}\left(\frac{2x(x+8)^{\frac{3}{2}}}{2(1)+3}-\frac{16(1)}{2(1)+3}I_0\right)=\ldots$ A full complete and correct method with limits applied to obtain a numerical value for I_2 (i.e. there should be no x's) **Dependent on both previous M marks**		**dd**M1
	$\int_0^{10}x^2\sqrt{(x+8)}\,\mathrm{d}x=\frac{97232}{105}\sqrt{2}$	Cao	A1
			(5)
	Useful information:		
	Expression without limits applied: $\boxed{I_2=\frac{2x^2(x+8)^{\frac{3}{2}}}{7}-\frac{64x(x+8)^{\frac{3}{2}}}{35}+\frac{1024(x+8)^{\frac{3}{2}}}{105}}$ This would imply the first 3 marks	**Expression with limits applied:** $\boxed{I_2=\frac{37872}{35}\sqrt{2}-\frac{16384}{105}\sqrt{2}}$	
	Value of I_1 $\boxed{I_1=\frac{2024}{15}\sqrt{2}}$		
			Total 11

(b)	Alternative by parts from scratch:		
	$I_2 = \int x^2\sqrt{(x+8)}\,\mathrm{d}x = \frac{2}{3}x^2(8+x)^{\frac{3}{2}} - \frac{4}{3}\int x(8+x)^{\frac{3}{2}}\,\mathrm{d}x$ M1: Correct first application of parts on I_2 A1: Correct expression		M1A1
	$= \frac{2}{3}x^2(8+x)^{\frac{3}{2}} - \frac{4}{3}\left(\frac{2}{5}x(8+x)^{\frac{5}{2}} - \int \frac{2}{5}(8+x)^{\frac{5}{2}}\,\mathrm{d}x\right)$ M1: Applies parts again		M1
	$= \frac{2}{3}x^2(8+x)^{\frac{3}{2}} - \frac{8}{15}x(8+x)^{\frac{5}{2}} + \frac{8}{15}\int (8+x)^{\frac{5}{2}}\,\mathrm{d}x$		
	$= \frac{2}{3}x^2(8+x)^{\frac{3}{2}} - \frac{8}{15}x(8+x)^{\frac{5}{2}} + \frac{16}{105}(8+x)^{\frac{7}{2}}$		
	$\left[\frac{2}{3}x^2(8+x)^{\frac{3}{2}} - \frac{8}{15}x(8+x)^{\frac{5}{2}} + \frac{16}{105}(8+x)^{\frac{7}{2}}\right]_0^{10} = \frac{200}{3}18^{\frac{3}{2}} - \frac{80}{15}18^{\frac{5}{2}} + \frac{16}{105}18^{\frac{7}{2}} - \frac{16}{105}8^{\frac{7}{2}}$ A fully complete and correct method including correct use of limits to obtain a numerical value for I_2 **Dependent on both previous M marks**		**dd**M1
	$= \frac{97232}{105}\sqrt{2}$	Cao	A1
			(5)
	Hybrid:		
	$I_1 = \int x\sqrt{(x+8)}\,\mathrm{d}x = \frac{2}{3}x(8+x)^{\frac{3}{2}} - \frac{2}{3}\int (8+x)^{\frac{3}{2}}\,\mathrm{d}x$ M1: Correct application of parts on I_1 A1: Correct expression		M1A1
	$I_2 = \frac{2x^2(x+8)^{\frac{3}{2}}}{2(2)+3} - \frac{16(2)}{2(2)+3}I_1$	Uses the given reduction formula on I_2	M1
	$I_1 = \int x\sqrt{(x+8)}\,\mathrm{d}x = \frac{2}{3}x(8+x)^{\frac{3}{2}} - \frac{4}{15}(8+x)^{\frac{5}{2}}$		
	$I_1 = \left[\frac{2}{3}x(8+x)^{\frac{3}{2}} - \frac{4}{15}(8+x)^{\frac{5}{2}}\right]_0^{10} = \frac{2024}{15}\sqrt{2}$		
	$\int_0^{10} x^2\sqrt{(x+8)}\,\mathrm{d}x = \left[\frac{2x^2(x+8)^{\frac{3}{2}}}{2(2)+3}\right]_0^{10} - \frac{32}{7}\times\frac{2024}{15}\sqrt{2} = ...$ M1: A complete method including correct use of limits **Dependent on both previous M marks**		**dd**M1
	$= \frac{97232}{105}\sqrt{2}$	Cao	A1
			(5)

Question Number	Scheme		Notes
6	$\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3}$ $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \mu\begin{pmatrix}1\\1\\3\end{pmatrix}$		
(a)	$\begin{pmatrix}2\\3\\-1\end{pmatrix} \neq k\begin{pmatrix}1\\1\\3\end{pmatrix}$	Shows lines are not parallel. If they say "different direction vectors", the direction vectors must be identified.	B1
	Examples of showing non-parallel: $\boxed{\frac{2}{1} \neq \frac{3}{1}}$, $\boxed{\begin{pmatrix}2\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}10\\-7\\-1\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix} (\text{allow} \neq 0)}$ $\boxed{\begin{pmatrix}2\\3\\-1\end{pmatrix} \bullet \begin{pmatrix}1\\1\\3\end{pmatrix} = 2 \neq \sqrt{14}\sqrt{11}}$ $\boxed{\begin{pmatrix}2\\3\\-1\end{pmatrix} \bullet \begin{pmatrix}1\\1\\3\end{pmatrix} = 2 = \sqrt{14}\sqrt{11}\cos\theta \Rightarrow \theta = 80.7^{\circ}}$		
	$\mathbf{i}:\ 1 + 2\lambda = -1 + \mu$ (1) $\mathbf{j}:\ 3\lambda = 4 + \mu$ (2) $\mathbf{k}:\ 2 - \lambda = 1 + 3\mu$ (3)		
	(1) and **(2)** yields $\lambda = 6,\ \mu = 14$ **(1)** and **(3)** yields $\lambda = -\frac{5}{7},\ \mu = \frac{4}{7}$ **(2)** and **(3)** yields $\lambda = \frac{13}{10},\ \mu = -\frac{1}{10}$	Attempts to solve a pair of equations to find at least one of either $\lambda = ...$ or $\mu = ...$	M1
	Checking **(3)**: $-4 \neq 43$ Checking **(2)**: $-\frac{15}{7} \neq \frac{32}{7}$ Checking **(1)**: $3.6 \neq -1.1$	Attempts to show a contradiction	M1
	So the lines are not parallel and do not intersect so the lines are skew	All complete and with no errors and conclusion. If they have already stated "not parallel" there is no need to repeat this.	A1
			(4)

	Alternative for the M marks:		
	(1) and **(2)** yields $\lambda = 6,\ \mu = 14$ **(1)** and **(3)** yields $\lambda = -\frac{5}{7},\ \mu = \frac{4}{7}$ **(2)** and **(3)** yields $\lambda = \frac{13}{10},\ \mu = -\frac{1}{10}$	Attempts to solve a pair of equations to find at least one of either $\lambda = ...$ or $\mu = ...$	M1
	Shows any two of **(1)** and **(2)** yielding $\lambda = 6$ **(1)** and **(3)** yielding $\lambda = -\frac{5}{7}$ **(2)** and **(3)** yielding $\lambda = \frac{13}{10}$ **or** shows any two of **(1)** and **(2)** yielding $\mu = 14$ **(1)** and **(3)** yielding $\mu = \frac{4}{7}$ **(2)** and **(3)** yielding $\mu = -\frac{1}{10}$	Attempts to show a contradiction	M1

Note that for (b) the only misinterpretations for Position we are allowing are:

$\begin{pmatrix}1\\2\\0\end{pmatrix}$ for $\begin{pmatrix}1\\0\\2\end{pmatrix}$ for the position of l_1 and $\begin{pmatrix}1\\-4\\-1\end{pmatrix}$ for $\begin{pmatrix}-1\\4\\1\end{pmatrix}$ for the position of l_2

But allow obvious slips or mis-copies of e.g. signs or elements if the intention is clear.

(b) Way 1	$\pm\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}=\pm\begin{pmatrix}10\\-7\\-1\end{pmatrix}$	Attempt cross product of direction vectors. If no method is shown, 2 components should be correct.	M1
		Correct vector	A1
	$\pm\left(\begin{pmatrix}1\\0\\2\end{pmatrix}-\begin{pmatrix}-1\\4\\1\end{pmatrix}\right)\bullet\pm\begin{pmatrix}10\\-7\\-1\end{pmatrix}=\pm(20+28-1)=\pm 47$	Attempt scalar product between the difference of the position vectors and their normal vector.	M1
	$d=\left\lvert\dfrac{\pm 47}{\sqrt{10^2+7^2+1^2}}\right\rvert=\dfrac{47}{\sqrt{150}}$	Correct completion. Divides their scalar product between the difference of the position vectors and their normal vector by the modulus of their vector product.	M1
		Any equivalent or awrt 3.84	A1
			(5)

(b) Way 2	$\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}=\begin{pmatrix}10\\-7\\-1\end{pmatrix}$	Attempt cross product of direction vectors	M1
		Correct vector	A1
	$\begin{pmatrix}10\\-7\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=8,\ \begin{pmatrix}10\\-7\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\4\\1\end{pmatrix}=-39$	Attempt equation of both planes	M1
	$d=\dfrac{8}{\sqrt{10^2+7^2+1^2}}-\dfrac{-39}{\sqrt{10^2+7^2+1^2}}=\dfrac{47}{\sqrt{150}}$	Correct completion	M1
		Any equivalent e.g. $\dfrac{47\sqrt{6}}{30}$ or awrt 3.84 but must be positive.	A1
			(5)

(b) Way 3	$\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}-\left[\begin{pmatrix}-1\\4\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\3\end{pmatrix}\right]=\begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}$ $\begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}=0,\ \begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}=0$ $2\lambda-11\mu=-1$ $14\lambda-2\mu=9$	Finds a general chord between the 2 lines and attempts the scalar product between this and the directions, sets = 0 to give 2 equations in 2 unknowns	M1
	$\lambda=\frac{101}{150},\ \mu=\frac{16}{75}$	Correct values	A1
	$\left(-\frac{59}{75},\frac{316}{75},\frac{41}{25}\right),\left(\frac{176}{75},\frac{303}{150},\frac{199}{150}\right)$ Or $\begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}=\begin{pmatrix}\frac{47}{15}\\-\frac{329}{150}\\-\frac{47}{150}\end{pmatrix}$	Uses their values to find the ends of the chord or substitutes into their chord vector	M1
	$d=\sqrt{\left(\frac{47}{15}\right)^2+\left(\frac{329}{150}\right)^2+\left(\frac{47}{150}\right)^2}=\frac{47\sqrt{6}}{30}$	Correct completion by finding the distance between their 2 points	M1
		Any equivalent e.g. $\frac{47\sqrt{6}}{30}$ or awrt 3.84	A1
			(5)

(b) Way 4	$\pm\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}=\pm\begin{pmatrix}10\\-7\\-1\end{pmatrix}$	Attempt cross product of direction vectors	M1
		Correct vector	A1
	$\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}-\left[\begin{pmatrix}-1\\4\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\3\end{pmatrix}\right]=\begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}$ $\begin{pmatrix}2+2\lambda-\mu\\-4+3\lambda-\mu\\1-\lambda-3\mu\end{pmatrix}=k\begin{pmatrix}10\\-7\\-1\end{pmatrix}\Rightarrow\begin{array}{c}2+2\lambda-\mu=10k\\-4+3\lambda-\mu=-7k\\1-\lambda-3\mu=-k\end{array}$ $\Rightarrow k=\frac{47}{150}$	Finds a common chord between the 2 lines and sets equal to a multiple of the normal vector to give 3 equations in 3 unknowns and solves to find a value for *k*	M1
	$d=\sqrt{\left(\frac{47}{15}\right)^2+\left(\frac{329}{150}\right)^2+\left(\frac{47}{150}\right)^2}=\frac{47\sqrt{6}}{30}$	Correct completion by finding the length of their vector	M1
		Any equivalent e.g. $\frac{47\sqrt{6}}{30}$ or awrt 3.84	A1
			(5)

(c)	$\begin{pmatrix}3\\8\\13\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\8\\11\end{pmatrix}$	Attempt another non-parallel vector in Π	M1
	$\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}2\\8\\11\end{pmatrix}=\begin{pmatrix}41\\-24\\10\end{pmatrix}$	Attempt cross product of two non-parallel vectors in the plane. If the method is not shown, at least 2 components should be correct. **Dependent on the first M mark.**	**d**M1
	$\begin{pmatrix}41\\-24\\10\end{pmatrix}\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=\ldots$ or $\begin{pmatrix}41\\-24\\10\end{pmatrix}\bullet\begin{pmatrix}3\\8\\13\end{pmatrix}=\ldots$	Attempt scalar product with a point in the plane. **Dependent on both previous method marks.**	**dd**M1
	$41x-24y+10z=61$	Any multiple but must be a Cartesian equation.	A1
			(4)
(c) Way 2	$\begin{pmatrix}3\\8\\13\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\8\\11\end{pmatrix}$	Attempt another vector in Π	M1
	$r=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\8\\11\end{pmatrix}$ or $\begin{matrix}x=1+2\lambda+2\mu\ (1)\\ y=3\lambda+8\mu\ (2)\\ z=2-\lambda+11\mu\ (3)\end{matrix}$	Forms the vector equation of the plane. **Dependent on the first M mark.**	**d**M1
	$(1)+2(3): x+2z=5+24\mu$ $(2)+3(3): 3z+y=6+41\mu$	Eliminates λ or μ. **Dependent on both previous method marks.**	**dd**M1
	$\dfrac{3z+y-6}{41}=\dfrac{x+2z-5}{24}$	Any correct equation but must be a correct Cartesian equation. Isw	A1
			(4)
			Total 13

Question Number	Scheme	Notes	Marks
7(a)	$ae = 3, \quad \frac{a}{e} = \frac{25}{3}$ or $ae = \pm 3, \quad \frac{a}{e} = \pm\frac{25}{3}$	Correct equations. (Ignore the use of + or – throughout)	B1
	$e^2 = \frac{9}{25}$ and $a^2 = 25$	Solves to find a or a^2 and e or e^2	M1
	$b^2 = a^2\left(1-e^2\right) \Rightarrow b^2 = 25\left(1-\frac{9}{25}\right) = 16$	Uses correct eccentricity formula to find b or b^2	M1
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1\left(\text{or } \frac{x^2}{5^2}+\frac{y^2}{4^2}=1\right)$	M1: Uses a correct ellipse formula and their a and b A1: Correct equation	M1A1
			(5)
(b)	$\frac{x^2}{25}+\frac{(mx+c)^2}{16}=1$	Substitutes for y. Allow in terms of a and b.	M1
	$16x^2+25\left(m^2x^2+2mcx+c^2\right)=400$ $\therefore \left(16+25m^2\right)x^2+50mcx+25\left(c^2-16\right)=0$*	Correct proof including sufficient intermediate working (at least one step) with no errors.	A1*
			(2)
(c)	$b^2-4ac=0 \Rightarrow (50mc)^2-4\left(16+25m^2\right)\left(25\left(c^2-16\right)\right)=0$ M1: Uses $b^2-4ac=0$ oe e.g. $b^2=4ac$ **with the given quadratic** (may be implied by their equation) Do not allow as part of an attempt to use the quadratic formula unless the discriminant is "extracted" and used = 0 A1: Correct equation (the "= 0" may be implied/appear later) The equation above scores M1A1		M1A1
	$c^2 = 25m^2+16$	cao	A1
			(3)
(d)	$x = \pm\frac{\sqrt{25m^2+16}}{m}, \ y = \sqrt{25m^2+16}$	Follow through their p and q. May be implied by their attempt at the triangle area.	B1ft
	Area OAB $(=T) = \frac{1}{2}\frac{\sqrt{25m^2+16}}{m}\sqrt{25m^2+16}$	Correct triangle area method (Allow ± area here)	M1
	$T = \frac{25m^2+16}{2m}$*	Correct area. (Must be positive)	A1*
			(3)
(d) Alt 1	$y = mx+c \Rightarrow y = c, \ x = \pm\frac{c}{m}$	Correct intercepts	B1
	Area OAB $(=T) = \frac{1}{2}\times c\times\frac{c}{m} = \frac{c^2}{2m}$	Correct triangle area method (Allow ± area here)	M1
	$T = \frac{25m^2+16}{2m}$*	Correct positive area. Must follow the final A1 in part (c) unless the work for part (c) is done in part (d).	A1*
			(3)

(e)	$\frac{dT}{dm} = \frac{25}{2} - \frac{8}{m^2} = 0 \Rightarrow m = \frac{4}{5}$ or $\frac{dT}{dm} = \frac{2m(50m) - 2(25m^2 + 16)}{4m^2} = 0 \Rightarrow m = \frac{4}{5}$ Solves $\frac{dT}{dm} = 0$ to obtain a value for m		M1
	$m = \frac{4}{5} \Rightarrow T = 20$	cao	A1
			(2)
	Alternative for (e)		
	$T = \frac{25m^2 + 16}{2m} = \frac{(5m-4)^2 + 40m}{2m}, (5m-4)^2 = 0 \Rightarrow T = \frac{40m}{2m}$ Writes T as $\frac{(5m-4)^2 + ...}{2m}$ and realises minimum when $(5m - 4) = 0$		M1
	$T = 20$	cao	A1
			(2)
			Total 15

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